A GLOBAL THEORY OF CONFORMAL FINSLER GEOMETRY

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Dedicated to the memory of Prof. Dr. A. TAMIM

Abstract. The aim of the present paper is to establish *a global investigation* of conformal changes in Finsler geometry. Under this change, we obtain the relationships between some geometric objects associated to (M, L) and the corresponding objects associated to $(M, \tilde{L})$, $\widetilde{L} = e^{\sigma(x)}L$ being the Finsler conformal transformation. We have found explicit global expressions relating the two associated Cartan connections ∇ and $\widetilde{\nabla}$, the two associated Berwald connections D and $\widetilde{D}$ and the two associated Barthel connections Γ and $\widetilde{\Gamma}$. The relationships between the corresponding curvature tensors have been also found. The relations thus obtained lead in turn to several interesting results.

Among the results obtained, is a characterization of conformal changes, a characterization of homotheties, some conformal invariants and conformal σ-invariants. In addition, several useful identities have been found.

Although our treatment is entirely global, the local expressions of the obtained results, when calculated, coincide with the existing classical local results. [1]

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Introduction

Studying Finsler geometry one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

In Riemannian geometry there is a canonical linear connection on the manifold M, whereas in Finsler geometry there is a corresponding canonical linear connection due to E. Cartan. However, this is not a connection on M but in $T(\mathcal{T}M)$, the tangent bundle of $\mathcal{T}M$ (*the Klein-Grifone approach*), or in $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \longrightarrow M$ (*the pullback approach*).

The infinitesimal transformations in Finsler geometry (such as conformal [8], projective [4], semi-projective [25], $\beta-$changes [20], conformal β-changes [1], ... etc.) play an important role not only in differential geometry but also in application to other branches of science, especially in the process of geometrization of physical theories. Conformal changes have been initiated by M. S. Kneblman [13] and have been investigated by many authors [8], [9], [10], [18],...etc. Almost all known result concerning these changes are local ones. The global results are very few in the literature. In this paper we present *a global theory* of conformal changes in Finsler geometry

The most well-known and widely used approaches to GLOBAL Finsler geometry are the Klein-Grifone (KG-) approach (cf. [6], [7], [12] and [25]) and the pull-back (PB-) approach (cf. [2], [3], [5], [14] and [21]). The universe of the first approach is the vector bundle $\pi_{TM} : TTM \longrightarrow TM$, whereas the universe of the second is the vector bundle $P : \pi^{-1}(TM) \longrightarrow TM$. Each of the two approaches has its own geometry which differs significantly from the geometry of the other (in spite of the existence of some links between them). Each also has its advantages and disadvantages. For example, the KG-formalism is an elegant one and easily manipulated, but the geometric aspects of many of its objects are not clarified. On the other hand, the PB-formalism is similar to and guided by Riemannian geometry, but is somewhat difficult to manipulate.

Most of the geometers, when treating Finsler geometry from a GLOBAL standpoint, follow **uniquely** one of the above mentioned approaches. We proceed here differently. We establish a global theory of conformal Finsler geometry within the PB-approach, making simultaneous use of the KG-approach. This has been done via certain links we have found between both approaches. This "double approach" enables us to overcome several difficulties and to continue our development. For instance, without the insertion of the KG-approach, we were unable to find the relation between the Cartan connection and its conformal transform, a relation which is fundamental for the present work. This shows that these two approaches are not alternatives but rather complementary.

Let (M, L) and $(M, \widetilde{L})$ be two conformal Finsler manifolds, where the conformal transformation is given by $L \longrightarrow \widetilde{L} = e^{\sigma(x)}L$. We have found explicit global expressions relating the two associated Cartan connections ∇ and $\widetilde{\nabla}$, the two associated Berwald connections D and $\widetilde{D}$ and the two associated Barthel connections Γ and $\widetilde{\Gamma}$. The relationships between the corresponding curvature tensors have also been found. The relations thus obtained lead in turn to several interesting results.

Among the results obtained, is a characterization of conformal changes, a characterization of homotheties, some conformal invariants and conformal σ-invariants. In addition, several useful identities have been found.

The last section of the paper presents an application of some of the results obtained. It deals with geodesics and Jacobi fields in the context of global Finsler geometry.

It should finally be noted that, although our treatment is entirely global, the local expressions of the obtained results, when calculated, coincide with the classical local results of Hashiguchi, Matsumoto and others. For the sake of completeness an appendix, concerning the local expressions of the most important geometric objects treated, is included.

1. Fundamentals of the pull-back formalism

In this section we give a brief account of the basic concepts of the pullback formalism necessary for this work. For more details refer to [2], [3], [5], [14] and [21]. We make the general assumption that all geometric objects we consider are of class C^∞. The following notation will be used throughout this paper:
M: a real differentiable manifold of finite dimension n and of class C^∞,
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \longrightarrow M$: the tangent bundle of M,
$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π,
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(M)$-module of differentiable sections of $\pi^{-1}(TM)$,
i_X : interior product with respect to $X \in \mathfrak{X}(M)$,
df : the exterior derivative of f,
$d_L := [i_L, d]$, i_L being the interior derivative with respect to the vector form L.

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$. The lift to $\pi^{-1}(TM)$ of a vector field X on M is the π-vector field $\overline{X}$ defined by $\overline{X}(u) = (u, X(\pi(u)))$. The lift to $\pi^{-1}(TM)$ of a 1-form ω on M is the π-form $\overline{\omega}$ defined by $\overline{\omega}(u) = (u, \omega(\pi(u)))$.

The tangent bundle $T(\mathcal{T}M)$ is related to the pullback bundle $\pi^{-1}(TM)$ by the short exact sequence

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho = (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) = j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J = \gamma o \rho$ is called the natural almost tangent structure of TM. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma o \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let ∇ be a linear connection (or simply a connection) in the pullback bundle $\pi^{-1}(TM)$. We associate to ∇ the map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection (or the deflection) map of ∇. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u . The connection ∇ is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M.$$

If M is endowed with a regular connection, then the vector bundle maps

$$\begin{aligned} \gamma : \pi^{-1}(TM) &\longrightarrow V(\mathcal{T}M), \\ \rho|_{H(\mathcal{T}M)} : H(\mathcal{T}M) &\longrightarrow \pi^{-1}(TM), \\ K|_{V(\mathcal{T}M)} : V(\mathcal{T}M) &\longrightarrow \pi^{-1}(TM) \end{aligned}$$

are vector bundle isomorphisms. Let us denote $\beta = (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho o \beta = id_{\pi^{-1}(\mathcal{T}M)}, \qquad \beta o \rho = \begin{cases} id_{H(\mathcal{T}M)} & \text{on H(TM)} \\ 0 & \text{on V(TM)} \end{cases} \tag{1.1}$$

The classical torsion tensor $\mathbf{T}$ of the connection ∇ is defined by

$$\mathbf{T}(X,Y) = \nabla_X \rho Y - \nabla_Y \rho X - \rho[X,Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M)$$

The horizontal and mixed torsion tensors, denoted respectively by Q and T, are defined by

$$Q(\overline{X},\overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \;\; T(\overline{X},\overline{Y}) = \mathbf{T}(\gamma\overline{X},\beta\overline{Y}) \quad \forall \overline{X},\overline{Y} \in \mathfrak{X}(\pi(M)).$$

The classical curvature tensor $\mathbf{K}$ of the connection ∇ is defined by

$$\mathbf{K}(X,Y)\rho Z = -\nabla_X\nabla_Y\rho Z + \nabla_Y\nabla_X\rho Z + \nabla_{[X,Y]}\rho Z \quad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal, mixed and vertical curvature tensors, denoted respectively by R, P and S, are defined by

$$R(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z}, \quad S(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}.$$

Definition 1.1. *A Finsler manifold of dimension n is a pair (M, L), where M is a differentiable manifold of dimension n and $L : TM \longrightarrow \mathbb{R}$ is a map (called Lagrangian) such that:*

(a) *$L(X) \geqq 0$, for all $X \in \mathcal{T}M$, $L(0) = 0$,*

(b) *L is C^∞ on $\mathcal{T}M$, C^1 on TM,*

(c) *L is homogenous of degree 1 in the directional argument y: $\mathcal{C} \cdot L = L$,*

(d) *The π-form g of order two with components $g_{ij} := \frac{1}{2}\dot{\partial}_j\dot{\partial}_i L^2$ is positive definite. Hence, the $\pi-$form g defines a positive definite metric in $\pi^{-1}(TM)$*
(Here, $\dot{\partial}_i$ denotes partial differentiation with respect to the directional argument y^i).

Theorem 1.2. [22] *Let (M, L) be a Finsler manifold. There exists a unique regular connection ∇ in $\pi^{-1}(TM)$ such that*

(a) ∇ *is metric*: $\nabla g = 0$,

(b) *The horizontal torsion of* ∇ *vanishes*: $Q = 0$,

(c) *The mixed torsion* T *of* ∇ *satisfies* $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

Such a connection is called the Cartan connection associated to the Finsler manifold (M, L).

For the Cartan connection ∇, we have

$$Ko\gamma = id_{\pi^{-1}(\mathcal{T}M)}, \qquad \gamma oK = \begin{cases} id_{V(\mathcal{T}M)} & \text{on V(TM)} \\ 0 & \text{on H(TM)} \end{cases} \tag{1.2}$$

Then, from (1.1) and (1.2), we get

$$\beta o\rho + \gamma oK = id_{T(\mathcal{T}M)} \tag{1.3}$$

Hence, if we set $h := \beta o\rho$, $v := \gamma oK$, then every vector field $X \in \mathfrak{X}(\mathcal{T}M)$ can be represented uniquely in the form

$$X = hX + vX = \beta\rho X + \gamma KX \tag{1.4}$$

The maps h and v are the horizontal and vertical projectors associated to the Cartan connection ∇: $h^2 = h$, $v^2 = v$, $h + v = id_{\mathfrak{X}(\mathcal{T}M)}$, $voh = hov = 0$.

One can show that the torsion of the Cartan connection has the property that $T(\overline{X}, \overline{\eta}) = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$, and associated to T we have the

Definition 1.3. [22] *Let* ∇ *be the Cartan connection associated to* (M, L)*. The torsion tensor field* T *of the connection* ∇ *induces a* π*-tensor field of type* $(0, 3)$*, denoted again* T*, defined by*:

$$T(\overline{X}, \overline{Y}, \overline{Z}) = g(T(\overline{X}, \overline{Y}), \overline{Z}), \quad \textit{for all} \quad \overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$$

and a π*-form* C *defined by:*

$$C(\overline{X}) := \textit{Trace of the map} \quad \overline{Y} \longmapsto T(\overline{X}, \overline{Y}), \quad \textit{for all} \quad \overline{X} \in \mathfrak{X}(\pi(M)).$$

Definition 1.4. [22] *With respect to the Cartan connection* ∇ *associated to* (M, L)*, we have*

– *The horizontal and vertical Ricci tensors* Ric^h *and* Ric^v *are defined respectively by*:

$$Ric^h(\overline{X}, \overline{Y}) := \textit{Trace of the map} \quad \overline{Z} \longmapsto R(\overline{X}, \overline{Z})\overline{Y}, \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)),$$

$$Ric^v(\overline{X}, \overline{Y}) := \textit{Trace of the map} \quad \overline{Z} \longmapsto S(\overline{X}, \overline{Z})\overline{Y}, \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

– *The horizontal and vertical Ricci maps* Ric_0^h *and* Ric_0^v *are defined respectively by*:

$$Ric^h(\overline{X}, \overline{Y}) = g(Ric_0^h(\overline{X}), \overline{Y}), \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)),$$

$$Ric^v(\overline{X}, \overline{Y}) = g(Ric_0^v(\overline{X}), \overline{Y}), \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

– *The horizontal and vertical scalar curvatures* Sc^h *,* Sc^v *are defined respectively by*:

$$Sc^h := Tr(Ric_0^h), \qquad Sc^v := Tr(Ric_0^v),$$

where R and S are respectively the horizontal and vertical curvature tensors of ∇.

Definition 1.5. [6] *A spray on M is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that*

(a) $\rho o X = \overline{\eta}$,

(b) *X is homogenous of degree 2 in y: $[\mathcal{C}, X] = X$.*

Let $E := \frac{1}{2}L^2$ be the energy of the Lagrangian L. One can show that

$$d_J E(X) = g(\rho X, \overline{\eta}), \quad \text{for all} \quad X \in \mathfrak{X}(\mathcal{T}M), \tag{1.5}$$

and consequently, $g(\overline{\eta}, \overline{\eta}) = L^2$. One can also show that the exterior 2−form $\Omega := dd_J E$ on TM is nondegenerate. The form Ω is called the fundamental form [6].

Proposition 1.6. [12] *Let (M, L) be a Finsler manifold. The vector field $G \in \mathfrak{X}(TM)$ determined by $i_G\Omega = -dE$ is a spray, called the canonical spray associated to the energy E.*

One can show, in this case, that $G = \beta o \overline{\eta}$, and G is thus horizontal with respect to the Cartan connection ∇.

Theorem 1.7. [23] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D in $\pi^{-1}(TM)$ such that*

(a) *D is torsion free,*

(b) *The canonical spray $G = \beta o \overline{\eta}$ is horizontal with respect to D,*

(c) *The mixed curvature P of D satisfies: $P \otimes \overline{\eta} = 0$.*

Such a connection is called the Berwald connection associated to the Finsler manifold (M, L).

We terminate this section by some concepts and results concerning the Klein-Grifone approach. For more details refer to [6], [7], [12] and [25].

Definition 1.8. [6] *A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^o on TM, such that*

$$J\Gamma = J, \qquad \Gamma J = -J.$$

We have to note that Γ defines on TM an almost product structure ($\Gamma^2 = I$, where I is the identity on TM). The horizontal and vertical projectors h and v associated to Γ are defined by $h := \frac{1}{2}(I + \Gamma)$, $v := \frac{1}{2}(I - \Gamma)$. Thus Γ gives rise to the decomposition $TTM = H(TM) \oplus V(TM)$, where $H(TM) := Im\, h = Ker\, v$, $V(TM) := Im\, v = Ker\, h$. We have $Jh = J$, $hJ = 0$, $Jv = 0$, $vJ = J$. The torsion T of a nonlinear connection Γ is the vector 2-form on TM defined by $T := \frac{1}{2}[J, \Gamma]$. The curvature of a nonlinear connection Γ is the vector 2-form $\Re$ on TM defined by $\Re := -\frac{1}{2}[h, h]$.

Theorem 1.9. [6] *On a Finsler manifold (M, L), there exists a unique conservative nonlinear connection ($d_h E = 0$) with zero torsion. It is given by:*

$$\Gamma = [J, G],$$

where G is the canonical spray. Such a connection is called the canonical connection or the Barthel connection associated to (M, L).

In conclusion, on a Finsler manifold (M, L), there are canonically associated three connections; two of which are linear (the Cartan connection ∇ (Thm. 1.2) and the Berwald connection D (Thm. 1.7)) and the third is nonlinear (the Barthel connection Γ (Thm. 1.9)). These three connections have the same horizontal and vertical distributions.

2. Nonlinear connections associated to a regular linear connection

In this section we introduce two nonlinear connections associated to a given regular connection in $\pi^{-1}(TM)$. We also introduce a nonlinear connection naturally associated to the Cartan connection . The relation between these three nonlinear connections and the Barthel connection is obtained.

Proposition 2.1. *To each regular linear connection D in $\pi^{-1}(TM)$, there are associated two nonlinear connections*

$$\Gamma_1 := I - 2\varphi^{-1}o\tilde{\varphi}, \qquad \Gamma_2 := 2\beta o\rho - I,$$

where $\tilde{\varphi} := \gamma oK$, $\varphi := \tilde{\varphi}|_{V(TM)}$.

Moreover, the horizontal and vertical projectors associated to Γ_1 and Γ_2 are given respectively by:

$$h_1 = I - \varphi^{-1}o\tilde{\varphi}, \qquad v_1 = \varphi^{-1}o\tilde{\varphi}$$

$$h_2 = \beta o\rho, \qquad v_2 = I - \beta o\rho$$

Proof. Firstly, since $\varphi := \tilde{\varphi}|_{V(TM)} = V(TM) \longrightarrow V(TM)$ is an isomorphism on $V(TM)$, $\tilde{\varphi} = \varphi$ on $V(TM)$ and $\tilde{\varphi} = 0$ on $H(TM)$, it follows that

$$\varphi^{-1}o\tilde{\varphi} = \begin{cases} id_{V(TM)} & \text{on V(TM)} \\ 0 & \text{on H(TM)} \end{cases}$$

Now, Γ_1 is a vector 1-form on TM satisfying $J\Gamma_1 = Jo(I - 2\varphi^{-1}o\tilde{\varphi}) = J$ and $\Gamma_1 J = (I - 2\varphi^{-1}o\tilde{\varphi})oJ = -J$. Hence, Γ_1 is a nonlinear connection on M.

Secondly, Γ_2 is a vector 1-form on TM satisfying $J\Gamma_2 = (\gamma o\rho)o(2\beta o\rho - I) = J$ and $\Gamma_2 J = (2\beta o\rho - I)o(\gamma o\rho) = -J$. Hence, Γ_2 is a nonlinear connection on M. $\square$

Proposition 2.2. *Let (M, L) be a Finsler manifold. To the Cartan connection ∇, there is associated a nonlinear connection*

$$\Gamma := \beta o\rho - \gamma oK.$$

Moreover, the horizontal and vertical projectors of Γ and ∇ are the same.

Proof. The proof is straightforward and we omit it. $\square$

Theorem 2.3. *Let (M, L) be a Finsler manifold. If ∇ is the Cartan connection in $\pi^{-1}(TM)$, then the two nonlinear connection Γ_1 and Γ_2 associated to ∇ coincide and both equal to*

$$\Gamma = \beta o\rho - \gamma oK,$$

Moreover, Γ coincides with the Barthel connection associated to (M, L): $\Gamma = [J, G]$.

Proof. Since ∇ is the Cartan connection in $\pi^{-1}(TM)$, then $K=\gamma^{-1}$ on $V(TM)$. Therefore, $\Gamma_1 = I-2\varphi^{-1}o\tilde{\varphi} = I-2\gamma o K \overset{(1.3)}{=} I-2(I-\beta o\rho) = \Gamma_2 \overset{(1.3)}{=} \beta o\rho - \gamma o K = \Gamma$. Moreover, the nonlinear connection Γ is conservative. In fact, $d_hE(X) = i_h dE(X) = hX\cdot E$. But since $2E = g(\overline{\eta},\overline{\eta})$, by (1.5), and since ∇ is metric, then $\frac{1}{2}d_hE(X) = hX\cdot g(\overline{\eta},\overline{\eta}) = 2g(\overline{\eta},\nabla_{hX}\overline{\eta}) = 0$.
Finally, one can show that Γ is torsion-free. Hence, by Theorem 1.9, Γ coincides with the Barthel connection. $\square$

The above consideration enables us to express the Barthel connection Γ in different equivalent forms:

$$\Gamma = I-2\gamma oK = 2\beta o\rho - I = \beta o\rho - \gamma oK = [J,G]. \tag{2.1}$$

Note that the first three expressions of Γ belong to the PB-formalism, whereas the last expression belongs to the KG-formalism. Relations (2.1) establishes a useful link between the two formalisms.

Lemma 2.4. [24] *Under an arbitrary change $L\longrightarrow\tilde{L}$ of Finsler structures on M, let the corresponding Cartan connections ∇ and $\tilde{\nabla}$ be related by $\tilde{\nabla}_X\overline{Y} = \nabla_X\overline{Y}+\omega(X,\overline{Y})$. If we denote*

$$\left.\begin{array}{ll} A(\overline{X},\overline{Y}) := \omega(\gamma\overline{X},\overline{Y}), & B(\overline{X},\overline{Y}) := \omega(\beta\overline{X},\overline{Y}), \\ N(\overline{X}) := B(\overline{X},\overline{\eta}), & N_o := N(\overline{\eta}), \end{array}\right\} \tag{2.2}$$

then we have

(a) $\omega(\overline{X},\overline{Y}) = A(KX,\overline{Y}) + B(\rho X,\overline{Y})$,

(b) $A(\overline{X},\overline{\eta}) = 0$,

(c) $\tilde{K} = K + No\rho\,,\quad \tilde{\beta} = \beta - \gamma oN$.

By using Lemma 2.4 and Theorem 2.3 , we have

Proposition 2.5. *Under a change $L\longrightarrow\tilde{L}$ of Finsler structures on M, the corresponding Barthel connections Γ and $\tilde{\Gamma}$ are related by*

$$\tilde{\Gamma} = \Gamma - 2\text{\L}, \textit{ with } \ \text{\L} := \gamma oNo\rho. \tag{2.3}$$

Moreover, we have $\tilde{h} = h - \text{\L}$, $\tilde{v} = v + \text{\L}$.

Proposition 2.6. *The following assertion are equivalent*:

$$(a)\ N=0, \qquad (b)\ N_o = 0, \qquad (c)\ \tilde{\Gamma}=\Gamma.$$

Proof. $(a)\Longrightarrow(b)$ is trivial.
$(b)\Longrightarrow(c)$: If $N_o=0$, then, by Lemma 2.4(b), $\tilde{\beta}(\overline{\eta}) = \beta(\overline{\eta}) - \gamma(N_o) = \beta(\overline{\eta})$. Hence, $\tilde{G}=G$ and so $\tilde{\Gamma}=\Gamma$.
$(c)\Longrightarrow(a)$: If $\tilde{\Gamma}=\Gamma$, then both $\tilde{\Gamma}$ and Γ have the same horizontal distribution. This implies that $\tilde{\beta}=\beta$. Then, by Lemma 2.4(b), $\gamma oN = 0$; from which $N=0$. $\square$

Remark 2.7. *The map $\text{\L}$ is a vector 1-form on TM satisfying $\text{\L}(V(TM)) = 0$ and $\text{\L}(T(TM))\subset V(TM)$. Consequently, $\text{\L}^2 = 0$ and $\text{\L}$ is thus an almost tangent structure on M.*

3. Conformal change of Barthel connection and its curvature tensor

Definition 3.1. *Let (M, L) and $(M, \tilde{L})$ be two Finsler manifolds. The two associated metrics g and $\tilde{g}$ are said to be conformal if there exists a positive differentiable function $\psi(x, y)$ on TM such that $\tilde{g}(\overline{X}, \overline{Y}) = \psi(x, y)g(\overline{X}, \overline{Y})$. Equivalently, g and $\tilde{g}$ are conformal iff $\tilde{L}^2 = \psi(x, y)L^2$. In this case, the transformation $L \longrightarrow \tilde{L}$ is said to be a conformal transformation and the two Finsler manifold (M, L) and $(M, \tilde{L})$ are said to be conformal.*

Proposition 3.2. *Let (M, L) and $(M, \tilde{L})$ be two Finsler manifolds.The two associated metrics g and $\tilde{g}$ are conformal iff the factor of proportionality $\psi(x, y)$ is independent of the directional argument y.*

Proof. If $\psi(x, y) = \psi(x)$, then $\tilde{g} = \psi(x)g$ is clearly a Finsler metric on M, and so g and $\tilde{g}$ are conformal.

Conversely, let g and $\tilde{g}$ be conformal. Then there exists a positive differentiable function $\psi(x, y)$ such that $\tilde{g}(\overline{X}, \overline{Y}) = \psi(x, y)g(\overline{X}, \overline{Y})$. Setting $\overline{X} = \overline{Y} = \overline{\eta}$, taking into account the fact that $g(\overline{\eta}, \overline{\eta}) = 2E$, we get $\tilde{E} = \psi(x, y)E$. Then, $d_J\tilde{E}(X) = Ed_J\psi(x, y)(X) + \psi(x, y)d_JE(X)$ for all $X \in \mathfrak{X}(\mathcal{T}M)$. But since $d_JE(X) = g(\rho X, \overline{\eta})$ by (1.5), then $\tilde{g}(\rho X, \overline{\eta}) = Ed_J\psi(x, y)(X) + \psi(x, y)g(\rho X, \overline{\eta})$. Therefore, $Ed_J\psi = 0$, from which $d_J\psi = 0$, and so $\psi(x, y)$ is independent of y. $\square$

From now on, we write the conformal transformation in the form $\tilde{g} = e^{2\sigma(x)}g$, with $\sigma(x)$ a positive function of x alone.

Definition 3.3. *Let (M, L) and $(M, \tilde{L})$ be two conformal Finsler manifolds with $\tilde{g} = e^{2\sigma(x)}g$. A geometric object W is said to be conformally invariant* (resp. conformally σ-invariant) *if $\tilde{W} = W$* (resp. $\tilde{W} = e^{2\sigma(x)}W$).

We need the following definition for subsequent use:

Definition 3.4. *The vertical gradient of a function $f \in \mathfrak{F}(TM)$, denoted $grad_v f$, is the vertical vector field JX defined by*

$$df(Y) = \bar{g}(JX, JY), \quad \textit{for all} \;\; Y \in \mathfrak{X}(TM),$$

where $\bar{g}$ is the metric on $V(TM)$ defined by [6]

$$\bar{g}(JY, JZ) = \Omega(JY, Z), \quad \textit{for all} \;\; Y, Z \in \mathfrak{X}(TM).$$

In view of Proposition 2.5 and Proposition 2.1.5 of [17], taking the above definition into account, we get

Theorem 3.5. *Let (M, L) and $(M, \tilde{L})$ be conformal Finsler manifolds with $\tilde{g} = e^{2\sigma(x)}g$. The associated Barthel connections $\tilde{\Gamma}$ and Γ are related by*

$$\left.\begin{array}{rl} & \tilde{\Gamma} \; = \; \Gamma - 2\L, \\ \textit{where} & \L := d\sigma \otimes \mathcal{C} + \sigma_1 J - d_J E \otimes grad_v\sigma - EF = \gamma o N o \rho \end{array}\right\} \tag{3.1}$$

(with the same notation of Proposition 2.5), $\sigma_1 := d_G\sigma$ and $F := [J, grad_v\sigma]$. Consequently, $\tilde{h} = h - \L$, $\quad \tilde{v} = v + \L$ or equivalently, $\tilde{\beta} = \beta - \L o\beta$, $\quad \tilde{K} = K + K o \L$.

Proof. Since $\tilde{g} = e^{2\sigma(x)}g$, then, using the fact that $2E = g(\overline{\eta}, \overline{\eta})$ and that $\sigma(x)$ is independent of y, we get

$$\tilde{\Omega} = 2e^{2\sigma(x)} d\sigma \wedge i_{\mathcal{C}}\Omega + e^{2\sigma(x)}\Omega.$$

This, together with the relation $i_G\Omega = -dE$, imply that

$$\tilde{G} = G + 2(E\ grad_v\sigma - \sigma_1\mathcal{C}). \tag{3.2}$$

Consequently, since $\Gamma = [J, G]$, we finally get after some manipulation

$$\tilde{\Gamma} = \Gamma - 2\left\{d\sigma \otimes \mathcal{C} + \sigma_1 J - d_J E \otimes grad_v\sigma - EF\right\}. \ \Box$$

Note that the vector form Ł in (3.1) is expressed in the PB-formalism by the RHS and in the KG-formalism by LHS. Note also that the *local expressions* of (3.1) and (3.2) coincide with the usual local expressions found in [19], [8],...etc.

Corollary 3.6. *In the course of the proof of Theorem 3.5, we have shown that*:

(a) $\tilde{E} = e^{2\sigma(x)}E$,

(b) $\tilde{G} = G + 2(E\ grad_v\sigma - \sigma_1\mathcal{C})$,

(c) $\tilde{\Omega} = e^{2\sigma(x)}\Omega + 2e^{2\sigma(x)} d\sigma \wedge i_{\mathcal{C}}\Omega$.

Consequently, if the conformal change is a homothety ($\sigma =$ constant), then the canonical spray G is conformally invariant and the fundamental form Ω is conformally $\sigma-$invariant

Theorem 3.7. *Let* (M, L) *and* $(M, \tilde{L})$ *be conformal Finsler manifolds with* $\tilde{g} = e^{2\sigma(x)}g$. *The curvature tensors* $\tilde{\mathfrak{R}}$ *and* $\mathfrak{R}$ *of the associated Barthel connections* $\tilde{\Gamma}$ *and* Γ *are related by*

$$\tilde{\mathfrak{R}}(X, Y) = \mathfrak{R}(X, Y) - [ŁX, ŁY] - Ł[hX, hY] + \mathfrak{U}_{X,Y}\{v[hX, ŁY] + Ł[hX, ŁY]\}, \tag{3.3}$$

where $\mathfrak{U}_{X,Y}\Theta(X, Y) = \Theta(X, Y) - \Theta(Y, X)$.

Proof. The proof follows from the fact that $\mathfrak{R}(X, Y) = -v[hX, hY]$ (cf. [25]) and that $Ł[ŁX, ŁY] = 0$, taking Theorem 3.5 into account. $\Box$

4. Conformal change of Cartan connection and its curvature tensors

Lemma 4.1. *Let* (M, L) *be a Finsler manifold. Let* g *be the Finsler metric associated with* L *and let* ∇ *be the Cartan connection determined by the metric* g. *Then, the following relations hold*

(a) $2g(\nabla_{vX}\rho Y, \rho Z) = vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[Z, vX]) + g(\rho Z, \rho[vX, Y])$.

(b) $2g(\nabla_{hX}\rho Y, \rho Z) = hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y)$
$-g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hZ, hX]) + g(\rho Z, \rho[hX, hY])$.

Proof. If ∇ is a metric connection in $\pi^{-1}(TM)$ with nonzero torsion $\mathbf{T}$, one can show that ∇ is completely determined by the relation

$$
\begin{aligned}
2g(\nabla_X \rho Y, \rho Z) =& X \cdot g(\rho Y, \rho Z) + Y \cdot g(\rho Z, \rho X) - Z \cdot g(\rho X, \rho Y) \\
& - g(\rho X, \mathbf{T}(Y, Z)) + g(\rho Y, \mathbf{T}(Z, X)) + g(\rho Z, \mathbf{T}(X, Y)) \\
& - g(\rho X, \rho[Y, Z]) + g(\rho Y, \rho[Z, X]) + g(\rho Z, \rho[X, Y]).
\end{aligned}
\tag{4.1}
$$

Let ∇ be the Cartan connection, then (a) follows from (4.1) by replacing X, Y, Z by vX, hY, hZ respectively, taking into account the third condition of Theorem 1.2.

Similarly, (b) follows from (4.1) by replacing X, Y, Z by hX, hY, hZ respectively and using the second condition of Theorem 1.2. $\square$

It is worthy noting that the *local expressions* of (a) and (b) of the above lemma coincide with the usual local expression found in [19], [15],...etc.

Theorem 4.2. *If (M, L) and $(M, \tilde{L})$ are conformal Finsler manifolds, then the associated Cartan connections ∇ and $\tilde{\nabla}$ are related by:*

$$\tilde{\nabla}_X \overline{Y} = \nabla_X \overline{Y} + \omega(X, \overline{Y}), \tag{4.2}$$

where

$$
\begin{aligned}
\omega(X, \overline{Y}) :=& (hX \cdot \sigma(x))\overline{Y} + (\beta\overline{Y} \cdot \sigma(x))\rho X - g(\rho X, \overline{Y})\overline{P} \\
& - T(N\overline{Y}, \rho X) + T'(ŁX, \beta\overline{Y}),
\end{aligned}
\tag{4.3}
$$

$\overline{P}$ being a π-vector field defined by

$$g(\overline{P}, \rho Z) = hZ \cdot \sigma(x) \tag{4.4}$$

and T' being a 2-form on TM, with values in $\pi^{-1}(TM)$, defined by

$$g(T'(ŁX, hY), \rho Z) = g(\mathbf{T}(ŁZ, hY), \rho X).$$

or equivalently by

$$g(T'(ŁX, hY), \rho Z) = g(T(N\rho Z, \rho Y), \rho X). \tag{4.5}$$

In particular,

(a) $\tilde{\nabla}_{\gamma\overline{X}} \overline{Y} = \nabla_{\gamma\overline{X}} \overline{Y},$

(b) $\tilde{\nabla}_{\tilde{\beta}\overline{X}} \overline{Y} = \nabla_{\beta\overline{X}} \overline{Y} - U(\beta\overline{X}, \overline{Y}),$

where $\quad U(\beta\overline{X}, \overline{Y}) = -\omega(\beta\overline{X}, \overline{Y}) + \nabla_{Ł\beta\overline{X}}\overline{Y} = -B(\overline{X}, \overline{Y}) + \nabla_{Ł\beta\overline{X}}\overline{Y}.$

Proof. Using Lemma 4.1(a) and Theorem 3.5, taking into account the fact that $\sigma = \sigma(x)$ is independent of y, we get

$$2g(\tilde{\nabla}_{\tilde{v}X} \rho Y, \rho Z) = 2g(\nabla_{vX} \rho Y, \rho Z) + g(\rho[ŁX, hY], \rho Z) + A_1(X, Y, Z), \tag{4.6}$$

where A_1 is the 3-form on TM defined by

$$A_1(X, Y, Z) := ŁX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[Z, ŁX]).$$

But since $\nabla g = 0$, then $A_1(X,Y,Z) = g(\nabla_{ŁX}\rho Y, \rho Z) + g(\mathbf{T}(ŁX, hY), \rho Z)$, and so

$$\tilde{\nabla}_{\tilde{v}X}\rho Y = \nabla_{vX}\rho Y + \nabla_{ŁX}\rho Y \tag{4.7}$$

Similarly, by Lemma 4.1(b) and Theorem 3.5, noting that $\rho[ŁX, ŁY] = 0$, we get

$$\begin{aligned} 2g(\tilde{\nabla}_{\tilde{h}X}\rho Y, \rho Z) =& 2g(\nabla_{hX}\rho Y, \rho Z) + 2hX \cdot \sigma(x) g(\rho Y, \rho Z) + \\ &+ 2hY \cdot \sigma(x) g(\rho X, \rho Z) - 2hZ \cdot \sigma(x) g(\rho X, \rho Y) \\ &- \{g(\rho[hX, ŁY], \rho Z) + g(\rho[ŁX, hY], \rho Z)\} \\ &+ A_2(X,Y,Z), \end{aligned} \tag{4.8}$$

where A_2 is the 3-form on TM defined by

$$\begin{aligned} A_2(X,Y,Z) =& - ŁX \cdot g(\rho Y, \rho Z) - ŁY \cdot g(\rho Z, \rho X) + ŁZ \cdot g(\rho X, \rho Y) \\ &+ g(\rho X, \rho[hY, ŁZ]) - g(\rho Y, \rho[hZ, ŁX]) + \\ &+ g(\rho X, \rho[ŁY, hZ]) - g(\rho Y, \rho[ŁZ, hX]). \end{aligned}$$

But since $\nabla g = 0$, then

$$\begin{aligned} A_2(X,Y,Z) =& - g(\nabla_{ŁX}\rho Y + \nabla_{ŁY}\rho X, \rho Z) - g(\mathbf{T}(ŁX, hY), \rho Z) \\ &- g(\mathbf{T}(ŁY, hX), \rho Z) + 2g(\mathbf{T}(ŁZ, hY), \rho X). \end{aligned}$$

Thus (4.8) reduces to

$$\begin{aligned} \tilde{\nabla}_{\tilde{h}X}\rho Y =& \nabla_{hX}\rho Y + (hX \cdot \sigma(x))\rho Y + (hY \cdot \sigma(x))\rho X - g(\rho X, \rho Y)\overline{P} \\ &- \mathbf{T}(ŁX, hY) - \mathbf{T}(ŁY, hX) + T'(ŁX, hY) - \rho[ŁX, hY]. \end{aligned} \tag{4.9}$$

Now, by (4.7) and (4.9), we get

$$\begin{aligned} \tilde{\nabla}_X\rho Y =& \nabla_X\rho Y + (hX \cdot \sigma(x))\rho Y + (hY \cdot \sigma(x))\rho X - g(\rho X, \rho Y)\overline{P} - \rho[ŁX, hY] \\ &- T(N\rho X, \rho Y) - T(N\rho Y, \rho X) + T'(ŁX, hY) + \nabla_{ŁX}\rho Y. \end{aligned} \tag{4.10}$$

Hence, the result follows from (4.10), making use of the identity $T(N\rho X, \rho Y) = \mathbf{T}(ŁX, hY) = \nabla_{ŁX}\rho Y - \rho[ŁX, hY]$. $\square$

It should be noted that the *local expressions* of the formulae (a) and (b) of the above theorem coincide with the usual local formulae, expressing the conformal change of Cartan connection, found in [19], [8], [11]...etc., where the *local expression* of Ł plays an important role.

Remark 4.3. *For all $X, Y \in \mathfrak{X}(TM)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$,*

– *the tensor ω satisfies the identity*: $\omega(\gamma\overline{X}, \overline{Y}) = 0$.

– *the tensor T' satisfies the identity*: $T'(ŁX, hY) = T'(ŁY, hX)$.

– *The map U satisfies the identity*: $U(\beta\overline{X}, \overline{\eta}) = 0$.

We have already some conformal invariants and conformal σ-invariants:

Proposition 4.4. *Let the Finsler manifolds (M,L) and $(M,\tilde{L})$ be conformal with $\tilde{g}=e^{2\sigma(x)}g$. Then*

(a) *If a π-tensor field W of type (1,p) is conformally invariant, then its trace $Tr(W)$ is conformally invariant.*

(b) *The map $\nabla_{\gamma\overline{X}}:\mathfrak{X}(\pi(M))\longrightarrow\mathfrak{X}(\pi(M)):\overline{Y}\longmapsto\nabla_{\gamma\overline{X}}\overline{Y}$ is conformally invariant. Consequently, if W is a conformally invariant π-tensor field, then so is $\nabla_{\gamma\overline{X}}W$.*

(c) *The vector π-form $\nabla\overline{X}:\mathfrak{X}(\pi(M))\longrightarrow\mathfrak{X}(\pi(M)):\overline{Y}\longmapsto\nabla_{\gamma\overline{Y}}\overline{X}$ is conformally invariant.*

(d) *The mixed torsion T of the Cartan connection is conformally invariant. Consequently, $\nabla_{\gamma\overline{X}}C$ is conformally invariant.*

(e) *The π-tensor $(dL\,o\,\gamma)/L$ is conformally invariant; or equivalently, the tensor d_JL/L is conformally invariant.*

(f) *The angular metric tensor $\hbar$ defined by $\hbar(\overline{X},\overline{Y})=g(\overline{X},\overline{Y})-\frac{1}{L^2}g(\overline{X},\overline{\eta})g(\overline{Y},\overline{\eta})$ is conformally σ-invariant.*

(g) *The tensor field $\mathbb{T}$ defined by*

$$\mathbb{T}(\overline{X},\overline{Y},\overline{Z},\overline{W})=(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z},\overline{W})+\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\frac{1}{L^2}g(\overline{X},\overline{\eta})T(\overline{Y},\overline{Z},\overline{W}).$$

is conformally σ-invariant.

Proof. Part (a) follows from the fact that $\{\overline{E}_i\}$ is an orthonormal basis with respect to g iff $\{e^{-\sigma(x)}\overline{E}_i\}$ is an orthonormal basis with respect to $\tilde{g}$. Parts (b) and (c) follow from Theorem 4.2(a). Part (d) follows from the definition of T and the fact that $\rho[\gamma\overline{X},\tilde{\beta}\overline{Y}]=\rho[\gamma\overline{X},\beta\overline{Y}]$. Finally, parts (e), (f) and (g) are obvious. $\Box$

Note that the tensor $\mathbb{T}$ defined in (g) is the so-called T-tensor, introduced locally by Matsumoto and Shibata [16]. Note also that some of the above invariants globalize some Hashiguchi invariants [8].

Some of the conformal invariants listed in Proposition 4.4 can be used to characterize conformality. For example, we have

Theorem 4.5. *Two Finsler metrics g and $\tilde{g}$ are conformal if, and only if,*

$$\frac{d_J\tilde{L}}{\tilde{L}}=\frac{d_JL}{L}.$$

Proof. Firstly, if g and $\tilde{g}$ are conformal, then $\frac{d_J\tilde{L}}{\tilde{L}}=\frac{d_JL}{L}$ by Proposition 4.4(e). Conversely, if $\frac{d_J\tilde{L}}{\tilde{L}}=\frac{d_JL}{L}$, then $\frac{\tilde{g}(\rho X,\overline{\eta})}{\tilde{L}^2}=\frac{g(\rho X,\overline{\eta})}{L^2}$, by Equation(1.5). Hence,

$$\tilde{g}(\rho X,\overline{\eta})=\phi(x,y)g(\rho X,\overline{\eta}),\ \text{where}\ \ \phi(x,y)=\frac{\tilde{L}^2}{L^2}. \tag{4.11}$$

For all $Y\in\mathfrak{X}(TM)$, we have $JY\cdot\tilde{g}(\rho X,\overline{\eta})=JY\cdot(\phi g(\rho X,\overline{\eta}))$; from which, since $\tilde{\nabla}g=\nabla g=0$,

$$\tilde{g}(\tilde{\nabla}_{JY}\rho X,\overline{\eta})+\tilde{g}(\rho X,\tilde{\nabla}_{JY}\overline{\eta})=d_J\phi(Y)g(\rho X,\overline{\eta})+\phi g(\nabla_{JY}\rho X,\overline{\eta})+\phi g(\rho X,\nabla_{JY}\overline{\eta}).$$

Using the definition of the Cartan torsion, Proposition 2.5 and the fact that $\nabla_{JX}\overline{\eta} = \rho X$, we get

$$\tilde{g}(\tilde{T}(\rho Y, \rho X), \overline{\eta}) + \tilde{g}(\rho[JY, hX], \overline{\eta}) + \tilde{g}(\rho X, \rho Y) = d_J\phi(x,y)(Y)g(\rho X, \overline{\eta})$$
$$+\phi(x,y)g(T(\rho Y, \rho X), \overline{\eta}) + \phi(x,y)g(\rho[JY, hX], \overline{\eta}) + \phi(x,y)g(\rho X, \rho Y).$$

Making use of Theorem 1.2, Equation (4.11) and the identity $T(\overline{X}, \overline{\eta}) = 0$, we conclude that

$$\tilde{g}(\rho X, \rho Y) = d_J\phi(x,y)(Y)g(\rho X, \overline{\eta}) + \phi(x,y)g(\rho X, \rho Y). \tag{4.12}$$

Setting $X = G$ in the above equation, we have

$$\tilde{g}(\rho Y, \overline{\eta}) = d_J\phi(x,y)(Y)g(\overline{\eta}, \overline{\eta}) + \phi(x,y)g(\rho Y, \overline{\eta}).$$

This, together with (4.11), yields $d_J\phi = 0$. Hence, by (4.12), $\tilde{g} = \phi g$, where ϕ is a (positive) function of x only. $\square$

Theorem 4.6. *Let (M, L) and $(M, \tilde{L})$ be two conformal Finsler manifolds. The curvature tensors of the associated Cartan connections ∇ and $\tilde{\nabla}$ are related by:*

$$\begin{aligned}\tilde{\boldsymbol{K}}(X,Y)\overline{Z} =& \boldsymbol{K}(X,Y)\overline{Z} - \mathfrak{U}_{X,Y}\{(\nabla_X B)(\rho Y, \overline{Z}) + B(\rho X, B(\rho Y, \overline{Z}))\\ &+ \frac{1}{2}B(\boldsymbol{T}(X,Y), \overline{Z})\} \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M).\end{aligned} \tag{4.13}$$

In particular,

(a) $\tilde{S}(\overline{X}, \overline{Y})\overline{Z} = S(\overline{X}, \overline{Y})\overline{Z}$.

(b) $\tilde{P}(\overline{X}, \overline{Y})\overline{Z} = P(\overline{X}, \overline{Y})\overline{Z} - V(\overline{X}, \overline{Y})\overline{Z}$,

where V is the vector π-form defined by

$$V(\overline{X}, \overline{Y})\overline{Z} = S(N\overline{X}, \overline{Y})\overline{Z} - (\nabla_{\gamma\overline{Y}}B)(\overline{X}, \overline{Z}) - B(T(\overline{Y}, \overline{X}), \overline{Z}). \tag{4.14}$$

(c) $\tilde{R}(\overline{X}, \overline{Y})\overline{Z} = R(\overline{X}, \overline{Y})\overline{Z} + H(\overline{X}, \overline{Y})\overline{Z}$,

where H is the vector π-form defined by

$$\begin{aligned}H(\overline{X}, \overline{Y})\overline{Z} =& S(N\overline{X}, N\overline{Y})\overline{Z} - \mathfrak{U}_{\overline{X},\overline{Y}}\{P(\overline{X}, N\overline{Y})\overline{Z} + (\nabla_{\beta\overline{X}}B)(\overline{Y}, \overline{Z})\\ &- (\nabla_{Ł\beta\overline{X}}B)(\overline{Y}, \overline{Z}) + B(\overline{X}, B(\overline{Y}, \overline{Z})) - B(T(N\overline{X}, \overline{Y}), \overline{Z})\}.\end{aligned} \tag{4.15}$$

Proof. By Theorem 4.2, we have

$$\tilde{\nabla}_X\tilde{\nabla}_Y\overline{Z} = \nabla_X\nabla_Y\overline{Z} + \omega(X, \nabla_Y\overline{Z}) + \nabla_X\omega(Y, \overline{Z}) + \omega(X, \omega(Y, \overline{Z})).$$

with similar expression for $\tilde{\nabla}_Y\tilde{\nabla}_X\overline{Z}$. Moreover,

$$\tilde{\nabla}_{[X,Y]}\overline{Z} = \nabla_{[X,Y]}\overline{Z} + \omega([X,Y], \overline{Z}).$$

The above formulae together with the definition of the curvature tensor $\mathbf{K}$ give rise to (4.13). Moreover, (a) follows from (4.13) by setting $X = \gamma\overline{X}$, $Y = \gamma\overline{Y}$, noting that $Ło\gamma = 0$, $hoŁ = 0$ and that $\mathbf{T}(\gamma\overline{X}, \gamma\overline{Y}) = 0$. Similarly, (b) follows from the same relation by setting $X = \tilde{\beta}\overline{X}$, $Y = \gamma\overline{Y}$, noting that $Ło\tilde{\beta} = Ło\beta$, $hoŁ = 0$ and that $\mathbf{T}(\gamma\overline{X}, Ł\tilde{\beta}\overline{Y}) = 0$. Finally, (c) follows from the same relation by setting $X = \tilde{\beta}\overline{X}$, $Y = \tilde{\beta}\overline{Y}$. $\square$

It is to be noted that the *local expressions* of (a), (b) and (c) of the above theorem coincide with the corresponding local expressions found in [9], [8], [11]...etc.

In view of the above theorem, we have

Proposition 4.7. *The following geometric objects are conformally invariant*:

(a) *The vertical curvature tensor S.*

(b) *The vertical Ricci tensor Ric^v.*

(c) *The scalar function $L^2 Sc^v$.*

(d) *The π-tensor field* $\mathbb{F}^v := \{Ric^v - \frac{Sc^v \hbar}{2(n-2)}\}$.

(e) *The vertical Einstein π-tensor field* $E^v := \{Ric^v - \frac{Sc^v}{2} g\}$.

Note that the tensor $\mathbb{F}^v$ in (d) is used in the definition of the special Finsler space S_4-like.

Proposition 4.8. *Assume that the π-tensor field H is traceless*: $Tr(H) = 0$. *Then, the following geometric object are conformally invariant*:

(a) *The horizontal Ricci tensor Ric^h.*

(b) *The scalar function $L^2 Sc^h$.*

(c) *The π-tensor field* $\mathbb{F}^h := \{Ric^h - \frac{Sc^h g}{2(n-1)}\}$.

(d) *The horizontal Einstein π-tensor field* $E^h := \{Ric^h - \frac{Sc^h}{2} g\}$.

Note that the tensor $\mathbb{F}^h$ in (c) is used in the definition of the special Finsler space R_3-like.

Lemma 4.9. *For all* $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, *we have*:

(a) $[\gamma\overline{X}, \gamma\overline{Y}] = \gamma(\nabla_{\gamma\overline{X}}\overline{Y} - \nabla_{\gamma\overline{Y}}\overline{X})$

(b) $[\gamma\overline{X}, \beta\overline{Y}] = -\gamma(P(\overline{Y}, \overline{X})\overline{\eta} + \nabla_{\beta\overline{Y}}\overline{X}) + \beta(\nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}))$

(c) $[\beta\overline{X}, \beta\overline{Y}] = \gamma(R(\overline{X}, \overline{Y})\overline{\eta}) + \beta(\nabla_{\beta\overline{X}}\overline{Y} - \nabla_{\beta\overline{Y}}\overline{X})$

Consequently, the horizontal distribution is completely integrable if, and only if, $R(\overline{X}, \overline{Y})\overline{\eta} = 0$.

Assume that $H(\overline{X}, \overline{Y})\overline{\eta} = 0$ for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$. Consequently, $\tilde{R}(\overline{X}, \overline{Y})\overline{\eta} = R(\overline{X}, \overline{Y})\overline{\eta}$, by Theorem 4.6(c). This, together with the above lemma, give rise to the following

Theorem 4.10. *Suppose that* $H(\overline{X}, \overline{Y})\overline{\eta} = 0$. *The horizontal distribution with respect to ∇ is completely integrable if, and only if, the horizontal distribution with respect to $\tilde{\nabla}$ is completely integrable.*

We terminate this section by the following

Theorem 4.11. *Under a Finsler conformal change $\tilde{g} = e^{2\sigma(x)}g$, the following assertions are equivalent*:

(a) *The π-tensor field N vanishes identically*: $N = 0$,

(b) *The π-vector field N_o vanishes identically*: $N_o = 0$,

(c) *The two associated Barthel connections coincide*: $\tilde{\Gamma} = \Gamma$,

(d) *The two associated Cartan connections coincide*: $\tilde{\nabla} = \nabla$,

(e) *The conformal transformation is a homothety*: $\sigma =$*constant.*

Proof. The equivalences $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$ have been established in Proposition 2.6. We shall now prove the sequence of implications $(e) \Longrightarrow (d) \Longrightarrow (a) \Longrightarrow (e)$.
$(e) \Longrightarrow (d)$: Let σ be constant. Then $d\sigma = 0$, $d_G\sigma = 0$, $grad_v\sigma = 0$ and $[J, grad_v\sigma] = 0$. This implies, by (3.1), that $Ł = 0$ (and so $N = 0$). Now, putting $\sigma =$constant, $Ł = 0$ and $N = 0$ in (4.3), we get $\omega(X, \overline{Y}) = 0$ and consequently $\tilde{\nabla} = \nabla$.
$(d) \Longrightarrow (a)$: If $\tilde{\nabla} = \nabla$, then, by (4.2) and (2.2), we have $0 = \omega(\beta\overline{X}, \overline{\eta}) = B(\overline{X}, \overline{\eta})$; from which $N = 0$.
$(a) \Longrightarrow (e)$: If $N = 0$, then $Ł = 0$ by (3.1). Now, we compute $\bar{g}(ŁX, \mathcal{C})$, where $\bar{g}$ is the metric defined in Definition 3.4:
$\bar{g}((d\sigma \otimes \mathcal{C})X, \mathcal{C}) = L^2 d\sigma(X)$.
$\bar{g}(\sigma_1 JX, \mathcal{C}) = d_J E(X)(G \cdot \sigma)$,
$\bar{g}((d_J E \otimes grad_v\sigma)X, \mathcal{C}) = d_J E(X)(G \cdot \sigma)$,
$\bar{g}(EF(X), \mathcal{C}) = E\{\bar{g}([JX, grad_v\sigma], \mathcal{C}) - \bar{g}(J[X, grad_v\sigma], \mathcal{C})\} = 0$,
Substituting the above expressions in $\bar{g}(ŁX, \mathcal{C}) = 0$, we get $d\sigma = 0$, from which σ is constant (provided that M is connected). $\square$

It should be noted that some important results of Hashiguchi [8] (*obtained in local coordinates*) are thus retrieved by some parts of the above theorem.

5. Conformal change of Berwald connection and its curvature tensors

Let (M, L) be a Finsler manifold. Let ∇ and D be respectively the Cartan connection and the Berwald connection associated to (M, L). Throughout, the entities associated to the Berwald connection will be marked by an asterisk "$*$".

Lemma 5.1. [23] *The horizontal maps β and β^*, associated to the Cartan connection ∇ and the Berwald connection D, coincide. Similarly, the deflection maps K and K^* coincide.*

The following result gives an explicit expression of the Berwald connection D in terms of the Cartan connection ∇.

Lemma 5.2. [23] *Let (M, L) be a Finsler manifold. The Cartan connection ∇ and the Berwald connection D are related by*

$$D_X\overline{Y} = \nabla_X\overline{Y} + P(\rho X, \overline{Y})\overline{\eta} - T(KX, \overline{Y}) \quad \forall\, X \in \mathfrak{X}(TM),\, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

In particular, we have

(a) $D_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y})$.

(b) $D_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + P(\overline{X},\overline{Y})\overline{\eta}$,

In what follows we assume that (M, L) and $(M, \tilde{L})$ are conformal. By Lemma 5.1 and Theorem 3.5, we get

Lemma 5.3. *Under a Finsler conformal change $L \longrightarrow \tilde{L} = e^{\sigma(x)}L$, we have*

$$\tilde{h^*} = h^* - Ł, \qquad \tilde{v^*} = v^* + Ł.$$

Theorem 5.4. *Under a Finsler conformal change $\tilde{g} = e^{2\sigma(x)}g$, we have*

$$\tilde{D}_X\overline{Y} = D_X\overline{Y} + \omega^*(X,\overline{Y}), \tag{5.1}$$

where $\omega^*(X,\overline{Y}) = K([\gamma\overline{Y}, Ł]X) + D_{ŁX}\overline{Y}$

In particular, we have

(a) $\tilde{D}_{\gamma\overline{X}}\overline{Y} = D_{\gamma\overline{X}}\overline{Y}$

(b) $\tilde{D}_{\tilde{\beta}\overline{X}}\overline{Y} = D_{\beta\overline{X}}\overline{Y} - \Psi(\beta\overline{X},\overline{Y})$,

where $\Psi(\beta\overline{X},\overline{Y}) = K([Ł,\gamma\overline{Y}]\beta\overline{X}) = -B^*(\overline{X},\overline{Y}) + D_{Ł\beta\overline{X}}\overline{Y}$.

Proof. The formula (5.1) follows from Theorems 4.2, 4.6 and Lemmas 5.2, 5.3:

$$\begin{aligned}
\tilde{D}_X\overline{Y} &= \tilde{\nabla}_X\overline{Y} + \tilde{P}(\rho X,\overline{Y})\overline{\eta} - \tilde{T}(\tilde{K}X,\overline{Y})\\
&= \nabla_X\overline{Y} - U(hX,\overline{Y}) + \nabla_{ŁX}\overline{Y} + P(\rho X,\overline{Y})\overline{\eta} - V(\rho X,\overline{Y})\overline{\eta}\\
&\quad - T(KX,\overline{Y}) - T(KŁX,\overline{Y})\\
&= D_X\overline{Y} - U(hX,\overline{Y}) - V(\rho X,\overline{Y})\overline{\eta} - T(KŁX,\overline{Y}) + \nabla_{ŁX}\overline{Y}\\
&= D_X\overline{Y} + \nabla_{Ł[hX,\gamma\overline{Y}]}\overline{\eta} - \nabla_{[ŁhX,\gamma\overline{Y}]}\overline{\eta} + D_{ŁX}\overline{Y}\\
&= D_X\overline{Y} + K([\gamma\overline{Y}, Ł]X) + D_{ŁX}\overline{Y} = D_X\overline{Y} + \omega^*(X,\overline{Y}).
\end{aligned}$$

The relations (a) and (b) follow from (5.1) by setting $X = \gamma\overline{X}$ and $X = \tilde{\beta}\overline{X}$ respectively. $\square$

We have to note that the *local expressions* of (a) and (b) can be found in [8], [9], [19] ...etc.

In view of the above theorem, we have

Proposition 5.5. *The π-tensor field ω^* has the properties*:

(a) $\omega^*(X,\rho Y) = \omega^*(Y,\rho X)$.

(b) $\omega^*(\gamma\overline{X},\overline{Y}) = 0$.

(a) $\omega^*(\tilde{\beta}\overline{X},\overline{Y}) = \omega^*(\beta\overline{X},\overline{Y})$.

Moreover, The map Ψ has the property: $\Psi(\beta\overline{X},\overline{\eta}) = 0$.

Theorem 5.6. *Under a Finsler conformal change $\tilde{g} = e^{2\sigma(x)}g$, we have*

$$\tilde{\boldsymbol{K}}^*(X,Y)\overline{Z} = \boldsymbol{K}^*(X,Y)\overline{Z} - \mathfrak{U}_{X,Y}\{(\nabla_X B^*)(\rho Y,\overline{Z}) + B^*(\rho X, B^*(\rho Y,\overline{Z})\} \tag{5.2}$$

for all $X, Y \in \mathfrak{X}(TM)$ and $\overline{Z} \in \mathfrak{X}(\pi(M))$.
In particular,

(a) $\tilde{S^*}(\overline{X},\overline{Y})\overline{Z} = S^*(\overline{X},\overline{Y})\overline{Z} = 0.$

(b) $\tilde{P^*}(\overline{X},\overline{Y})\overline{Z} = P^*(\overline{X},\overline{Y})\overline{Z} + (D_{\gamma\overline{Y}}B^*)(\overline{X},\overline{Z}).$

(c) $\tilde{R^*}(\overline{X},\overline{Y})\overline{Z} = R^*(\overline{X},\overline{Y})\overline{Z} + H^*(\overline{X},\overline{Y})\overline{Z},$

where H^* is the vector π-form defined by

$$\begin{aligned} H^*(\overline{X},\overline{Y})\overline{Z} =& \mathfrak{U}_{\overline{X},\overline{Y}}\{P^*(\overline{Y},N\overline{X})\overline{Z} - (D_{\beta\overline{X}}B^*)(\overline{Y},\overline{Z}) \\ &+ (D_{\L\beta\overline{X}}B^*)(\overline{Y},\overline{Z}) - B^*(\overline{X},B^*(\overline{Y},\overline{Z}))\}. \end{aligned}$$

Proof. Equation (5.2) follows from the definition of the curvature tensor $\mathbf{K}^*$, together with Theorem 5.4 and Lemma 5.3. Part (a) follows by setting $X = \gamma\overline{X}$ and $Y = \gamma\overline{Y}$ in (5.2), taking Proposition 5.5 into account. Relation (b) follows by setting $X = \tilde{\beta}\overline{X}$ and $Y = \gamma\overline{Y}$ in (5.2), taking Lemma 5.3 and Proposition 5.5 into account. Relation (c) follows by setting $X = \tilde{\beta}\overline{X}$ and $Y = \tilde{\beta}\overline{Y}$ in the same equation. $\Box$

The *local expressions* of (b) and (c) of the above theorem are the same as those found in [9], [19], [8],...etc.

Proposition 5.7. *Let the Finsler manifolds (M,L) and $(M,\tilde{L})$ be conformal.*
The following geometric objects are conformally invariant:
(a) *The map $D_{\gamma\overline{X}} : \mathfrak{X}(\pi(M)) \longrightarrow \mathfrak{X}(\pi(M)) : \overline{Y} \longmapsto D_{\gamma\overline{X}}\overline{Y}$.*

(b) *The vector π-form $D\overline{X} : \mathfrak{X}(\pi(M)) \longrightarrow \mathfrak{X}(\pi(M)) : \overline{Y} \longmapsto D_{\gamma\overline{Y}}\overline{X}$.*

Given that the π-form B^* is vertically parallel: $D_{\gamma\overline{X}}B^* = 0$, then
(c) *The mixed curvature tensor P^* is conformally invariant.*

Given that the π-tensor field H^* is traceless: $Tr(H^*) = 0$, then the following geometric objects are conformally invariant:
(d) *The horizontal Ricci tensor Ric^{*h}.*

(e) *The scalar function $L^2 Sc^{*h}$.*

(f) *The horizontal Einstein π-tensor field $E^{*h} := \{Ric^{*h} - \frac{Sc^{*h}}{2}g\}$.*

6. Application: Geodesics and Jacobi fields

In this section we present an application of some of the obtained results.

Let $c : I \longrightarrow M$ be a regular curve in M. The canonical lift of c is the curve $\hat{c}$ in $\mathcal{T}M$ defined by $\hat{c} : t \longmapsto dc/dt$. The lift of a vector field $X \in \mathfrak{X}(M)$ along c is the π-vector field along $\hat{c}$ defined by $\overline{X} : \hat{c}(t) \longmapsto (\hat{c}(t), X(c(t)))$. In particular, the velocity

vector field dc/dt along c is lifted to the π-vector field $\overline{dc/dt} := (dc/dt, dc/dt)$ along $\hat{c}$. Clearly, $\rho(d\hat{c}/dt) = \overline{dc/dt} = \overline{\eta}\,|_{\hat{c}(t)}$. A vector field X along a regular curve c in M is parallel along c with respect to the connection ∇ (or is ∇-parallel along c) if $D\overline{X}/dt = \nabla_{d\hat{c}/dt}\overline{X} = 0$, where D/dt is the covariant derivative operator associated with ∇, along $\hat{c}$. A regular curve c in M is a geodesic if the π-vector field $\frac{D}{dt}(\frac{\overline{dc}}{dt})$ vanishes identically. In this case, the vector field $d\hat{c}/dt$ along $\hat{c}$ is horizontal with respect to ∇.

In what follows we assume that (M, L) and $(M, \tilde{L})$ are conformal Finsler manifolds with $\tilde{g} = e^{2\sigma(x)}g$. From Theorem 4.2, the associated Cartan connections ∇ and $\tilde{\nabla}$ are related by

$$\tilde{\nabla}_X\overline{Y} = \nabla_X\overline{Y} + \omega(X, \overline{Y}), \tag{6.1}$$

where ω is given by (4.3).

Let D/dt and $\tilde{D}/dt$ be the associated covariant operators (along a curve $\hat{c}$ in $\mathcal{T}M$) with respect to the Cartan connections ∇ and $\tilde{\nabla}$ respectively. Then for every π-vector field $\overline{X}$ along $\hat{c}$, taking (6.1) into account, we get

$$\tilde{D}\overline{X}/dt = D\overline{X}/dt + \omega(d\hat{c}/dt, \overline{X}). \tag{6.2}$$

A direct consequence of (6.2) is the following

Lemma 6.1. *Let c be a regular curve in M and X a vector field along c. If X is ∇-parallel (resp. $\tilde{\nabla}$-parallel) along c, then a necessary and sufficient condition for X to be $\tilde{\nabla}$-parallel (resp. ∇-parallel) along c is that $\omega(d\hat{c}/dt, \overline{X}) = 0$.*

Theorem 6.2. *A necessary and sufficient condition for a geodesic c in (M, L) (resp. $(M, \tilde{L})$) to be a geodesic in $(M, \tilde{L})$ (resp. (M, L)) is that $B(\overline{\vartheta}, \overline{\vartheta}) = 0$, where $\overline{\vartheta} := \overline{\eta}|_{\hat{c}(t)}$ and B is the π-tensor field defined by (2.2).*

Proof. Let c be a regular curve in M and let $\hat{c}$ be its canonical lift to $\mathcal{T}M$. Then, using Equation (6.2) and the fact that $A(\overline{X}, \overline{\eta}) = 0$ (Lemma 2.4(a)), we obtain

$$\tilde{D}\overline{\vartheta}/dt = D\overline{\vartheta}/dt + \omega(\beta\rho(d\hat{c}/dt), \overline{\vartheta}) = D\overline{\vartheta}/dt + \omega(\beta\overline{\vartheta}, \overline{\vartheta}) = D\overline{\vartheta}/dt + B(\overline{\vartheta}, \overline{\vartheta}).$$

The result follows from the last relation noting that c is geodesic in (M, L) (resp. $(M, \tilde{L})$) iff $D\overline{\vartheta}/dt = 0$ (resp. $\tilde{D}\overline{\vartheta}/dt = 0$). $\square$

Definition 6.3. [24] *A vector field $\xi \in \mathfrak{X}(M)$ along a geodesic c in M is called a Jacobi field with respect to a regular connection ∇ in $\pi^{-1}(TM)$ if it satisfies the Jacobi differential equation*

$$D^2\overline{\xi}/dt^2 + R(\overline{\vartheta}, \overline{\xi})\overline{\vartheta} = 0,$$

where R is the h-curvature of ∇, $\overline{\xi}$ is the lift of ξ along c and $\overline{\vartheta} = \overline{\eta}|_{\hat{c}}$.

Theorem 6.4. *Let c be a geodesic in M and $\overline{\vartheta} = \overline{\eta}|_{\hat{c}(t)}$. Assume that $H(\overline{\vartheta}, \overline{X})\overline{\vartheta} = 0$ and that the π-tensor field $i_{\overline{\vartheta}}B$ vanishes. A vector field $\xi \in \mathfrak{X}(M)$ along c is a Jacobi field with respect to ∇ if, and only if, it is a Jacobi field with respect to $\tilde{\nabla}$.*

Proof. It should firstly be noted that a geodesic c in (M, L) is also a geodesic in $(M, \tilde{L})$ since $i_{\overline{\vartheta}}B = 0$. By hypothesis, we have

$$\tilde{R}(\overline{\vartheta}, \overline{X})\overline{\vartheta} = R(\overline{\vartheta}, \overline{X})\overline{\vartheta} \quad \forall\, \overline{X} \in \mathfrak{X}(\pi(M)). \tag{6.3}$$

Now, let $\overline{\xi}$ be the lift of the vector field ξ along c. Putting $\overline{X} = \overline{\xi}$ in (6.2) and noting that $\beta o\rho + \gamma oK = id_{\mathfrak{X}(\mathcal{T}M)}$, we get

$$\begin{aligned}\tilde{D}\overline{\xi}/dt &= D\overline{\xi}/dt + \omega(\beta\rho(d\hat{c}/dt), \overline{\xi}) + \omega(\gamma K(d\hat{c}/dt), \overline{\xi}) \\ &= D\overline{\xi}/dt + B(\overline{\vartheta}, \overline{\xi}) + A(K(d\hat{c}/dt), \overline{\xi}).\end{aligned}$$

Moreover, since c is a geodesic and since $B(\overline{\vartheta}, \overline{X}) = 0$, it follows that $\tilde{D}\overline{\xi}/dt = D\overline{\xi}/dt$. Consequently,

$$\tilde{D}^2\overline{\xi}/dt^2 = D^2\overline{\xi}/dt^2. \tag{6.4}$$

According to Definition 6.3, the result follows from (6.3) and (6.4). $\square$

Concluding remarks

• A global theory of conformal Finsler geometry is established. Some known results are generalized and several new results are obtained.
• It is shown that the Pull-back formalism and the Klein-Grifone formalism are not alternatives but rather complementary.
• Although our treatment is entirely global, the local expressions of the obtained results, when calculated, coincide with the known classical local results (See the Appendix).
• The conformal change of different types of special Finsler spaces is not treated in the present work. It merits a separate study that we are currently in the process of preparing, and it will be the object of a forthcoming paper.
• The most important and well known connections in Finsler geometry are the Cartan, Berwald and Barthel connections. However, there are other connections of particular importance, such as Hashiguchi and Chern (Rund) connections. Such connections are not treated here. We are currently investigating these connections (and their conformal transforms) from a global standpoint, in the hope to build, with the material we have, a global theory of Finsler geometry as complete as possible.

Appendix. Local formulae

For the sake of completeness, we present in this appendix a brief and concise survey of the local expressions of the most important geometric objects treated in the paper.

Let $(U, (x^i))$ be a system of local coordinates on M and $(\pi^{-1}(U), (x^i, y^i))$ the associated system of local coordinates on $\mathcal{T}M$. We use the following notations:
$(\partial_i) := (\frac{\partial}{\partial x^i})$: the natural basis of T_xM, $x \in M$,
$(\dot{\partial}_i) := (\frac{\partial}{\partial y^i})$: the natural basis of $V_u(\mathcal{T}M)$, $u \in \mathcal{T}M$,
$(\partial_i, \dot{\partial}_i)$: the natural basis of $T_u(\mathcal{T}M)$,
$(\overline{\partial}_i)$: the natural basis of the fiber over u in $\pi^{-1}(TM)$ ($\overline{\partial}_i$ is the lift of ∂_i at u).

To a Finsler manifold (M, L), we associate the geometric objects:
$g_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = \dot{\partial}_i\dot{\partial}_j E$: the Finsler metric tensor,
G^h: the components of the canonical spray,
$G^h_i := \dot{\partial}_i G^h$,
$G^h_{ij} := \dot{\partial}_j G^h_i = \dot{\partial}_j\dot{\partial}_i G^h$,
$(\delta_i) := (\partial_i - G^h_i\dot{\partial}_h)$: the basis of $H_u(\mathcal{T}M)$ adapted to G^h_i,
$(\delta_i, \dot{\partial}_i)$: the basis of $T_u(\mathcal{T}M) = H_u(\mathcal{T}M) \oplus V_u(\mathcal{T}M)$ adapted to G^h_i.

We have:
$\gamma(\overline{\partial}_i) = \dot{\partial}_i$,
$\rho(\partial_i) = \overline{\partial}_i, \quad \rho(\dot{\partial}_i) = 0, \quad \rho(\delta_i) = \overline{\partial}_i$,
$\beta(\overline{\partial}_i) = \delta_i$,
$J(\partial_i) = \dot{\partial}_i, \quad J(\dot{\partial}_i) = 0, \quad J(\delta_i) = \dot{\partial}_i$,
$h = \beta o \rho = dx^i \otimes \partial_i - G^i_j \, dx^j \otimes \dot{\partial}_i, \quad v = \gamma o K = dy^i \otimes \dot{\partial}_i + G^i_j \, dx^j \otimes \dot{\partial}_i.$

We define:
$\gamma^h_{ij} := \frac{1}{2} g^{h\ell}(\partial_i \, g_{\ell j} + \partial_j \, g_{i\ell} - \partial_\ell \, g_{ij})$,
$C^h_{ij} := \frac{1}{2} g^{h\ell}(\dot{\partial}_i \, g_{\ell j} + \dot{\partial}_j \, g_{i\ell} - \dot{\partial}_\ell \, g_{ij}) = \frac{1}{2} g^{h\ell} \dot{\partial}_i \, g_{\ell j}$ (cf. Lemma 4.1(a)),
$\Gamma^h_{ij} := \frac{1}{2} g^{h\ell}(\delta_i \, g_{\ell j} + \delta_j \, g_{i\ell} - \delta_\ell \, g_{ij})$ (cf. Lemma 4.1(b)).
Then, we have:

- The canonical spray G: $G^h = \frac{1}{2} \gamma^h_{ij} \, y^i y^j$.
- The Barthel connection Γ: $G^h_i = \dot{\partial}_i G^h = G^h_{ij} \, y^i = \Gamma^h_{ij} \, y^j = \Gamma^h_{io}$ (cf. Equation (2.1)).
- The Cartan connection $C\Gamma$: $(G^h_i, \ \Gamma^h_{ij}, \ C^h_{ij})$.
- The Berwald connection $B\Gamma$: $(G^h_i, \ G^h_{ij}, \ 0)$.

We also have $G^h_{ij} = \Gamma^h_{ij} + C^h_{ij|k} \, y^k = \Gamma^h_{ij} + C^h_{ij|o}$, where the the stroke " $|$ " denotes the horizontal Cartan covariant derivative (cf. Lemma 5.2(b)).

Under a conformal change $g_{ij} \longrightarrow \tilde{g}_{ij} = e^{2\sigma(x)} g_{ij}$, we have the following expressions for the relationships between various geometric objects and their conformal transforms:

- ***Canonical spray***

$\tilde{G}^h = G^h - B^h$,
where $B^h := (E g^{hj} - y^h y^j)\sigma_j; \quad \sigma_j := \partial_j \sigma$.
(local expression of Equation (3.2)).

- ***Barthel connection***

$\tilde{G}^h_j = G^h_j - B^h_j$,
where $B^h_j := \dot{\partial}_j B^h = y_j \sigma^h - \delta^h_j \sigma_o - y^h \sigma_j - L^2 C^h_j, \ \sigma_o := \sigma_i y^i, \ \sigma^h := g^{hj}\sigma_j, \ C^i_j := C^{ir}_j \sigma_r$ and $C^{ir}_j := g^{rk} C^i_{jk}$.
(local expression of Equation (3.1)).

- ***Barthel curvature tensor***

$\tilde{\Re}^h_{ij} = \Re^h_{ij} + H^h_{ij}$,
where $H^h_{ij} := -\mathfrak{U}_{ij}\{B^h_{i|j} + (B^h_{im} - P^h_{im})B^m_j\}; \quad P^h_{im} := C^h_{im|o}$.
(local expression of Equation (3.3)).

- ***Cartan connection***

– $\tilde{C}^h_{ij} = C^h_{ij}$.
– $\tilde{\Gamma}^h_{ij} = \Gamma^h_{ij} - U^h_{ij}$,
where $U^h_{ij} := g_{ij}\sigma^h - \delta^h_i \sigma_j - \delta^h_j \sigma_i - C^h_{im} B^m_j - C^h_{jm} B^m_i + g^{hr} C_{ijm} B^m_r$.
(local expressions of Theorem 4.2(a), (b)).

- ***Cartan Curvature tensors***

– $\tilde{S}^h_{kij} = S^h_{kij}$.
– $\tilde{P}^h_{kij} = P^h_{kij} - V^h_{kij}$,
where $V^h_{kij} := 2B^m_i S^h_{kjm} + \dot{\partial}_j A^h_{ki} - U^h_{im} C^m_{kj} + U^m_{ki} C^h_{jm}$; $A^h_{ij} = U^h_{ij} + C^h_{im} B^m_j$.
– $\tilde{R}^h_{kij} = R^h_{kij} + H^h_{kij}$,
where $H^h_{kij} := 2S^h_{kml} B^m_i B^l_j - \mathfrak{U}_{ij}\{A^h_{ki|j} + B^m_j \dot{\partial}_m A^h_{ki} + U^m_{kj} U^h_{im} - B^m_j P^h_{kim}\}$.
(The local expressions of Theorem 4.6(a), (b), (c)).

- ***Berwald connection***

– $\tilde{C}^{*h}_{ij} = C^{*h}_{ij} = 0$,
where $D_{\dot{\partial}_i} \overline{\partial}_j =: C^{*h}_{ij}\, \overline{\partial}_h$.
– $\tilde{G}^h_{ij} = G^h_{ij} - \Psi^h_{ij}$,
where $D_{e_i} \overline{\partial}_j =: G^h_{ij}\, \overline{\partial}_h$, $\quad \Psi(e_i, \overline{\partial}_j) =: \Psi^h_{ij}\, \overline{\partial}_h = \dot{\partial}_j B^h_i\, \overline{\partial}_h =: B^h_{ij}\, \overline{\partial}_h$.
(The local expressions of Theorem 5.4(a), (b)).

- ***Berwald Curvature tensors***

– $\tilde{S}^{*h}_{kij} = S^{*h}_{kij} = 0$.
– $\tilde{P}^{*h}_{kij} = P^{*h}_{kij} - V^{*h}_{kij}$,
where $V^{*h}_{kij} := \dot{\partial}_j B^h_{ki}$ (note that $P^{*h}_{kij} = \dot{\partial}_j G^h_{ki}$).
– $\tilde{R}^{*h}_{kij} = R^{*h}_{kij} + H^{*h}_{kij}$,
where $H^{*h}_{kij} := \mathfrak{U}_{ij}\{(\dot{\partial}_m G^h_{ik}) B^m_j - B^h_{ik(j)} - (\dot{\partial}_m B^h_{ik}) B^m_j - B^h_{im} B^m_{kj}\}$.
The parentheses "( )" denote the horizontal Berwald covariant derivative.
(The local expression of Theorem 5.6(a), (b), (c)).

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